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Theory of Non-Degenerated Oscillatory Flows

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The aim of this paper is to derive the averaged governing equations for non-degenerated oscillatory flows, in which the magnitudes of mean velocity and oscillating velocity are similar. We derive the averaged equations for a scalar passive admixture, for a vectorial passive admixture (magnetic field in kinematic MHD), and for vortex dynamics. The small parameter of our asymptotic theory is the inverse dimensionless frequency $1/\sigma$. Our mathematical approach combines the two-timing method, distinguished limits, and the use of commutators to simplify calculations. This approach produces recurrent equations for both the averaged and oscillating parts of unknown fields. We do not use any physical or mathematical assumptions (except the most common ones) and present calculations for the first three (zeroth, first, and second) successive approximations. In all our examples the averaged equations exhibit the universal structure: the Reynolds-stress-type terms (or the cross-correlations) are transformed into drift velocities, pseudo-diffusion, and two other terms reminiscent of Moffatt's mean-fields in turbulence. In particular, the averaged motion of a passive scalar admixture is described only by a drift and pseudo-diffusion. The averaged equations for a passive vectorial admixture and for vortex dynamics possess two mean-field terms, additional to pseudo-diffusion. It is remarkable that all mean-field terms (including pseudo-diffusion) are expressed by invariant operators (Lie-derivatives) which measure the deviation of some tensors from their 'frozen-in' values. Some physical assumptions and the results can be build upon obtained averaged equations. Our physical interpretation suggests purely kinematic nature of pseudo-diffusion.

1. Introduction

Oscillatory flows play important roles in all branches of fluid dynamics, especially in wave theory, in biological and medical fluid mechanics, in geophysical and astrophysical fluid dynamics. If a chosen fluid flow is not oscillating, it can be easily made so by adding some oscillations to boundary conditions, *etc.* There is no unique definition of an oscillatory flow. In this paper we suggest that such a flow contains velocity oscillations with frequency higher than inverse characteristic times of all other co-existing motions. It can be expressed as $\sigma \gg 1$, where σ is dimensionless frequency of oscillations. Our term *a non-degenerated oscillatory flow* means that the mean and oscillatory velocity are similar in magnitudes. This class of flows is rather restrictive: in many applications the oscillating part of velocity is dominating (hence, the flows are degenerated), see Stokes (1847), Craik (1985), Magar & Pedley (2005), Riley (2001), Buhler (2009), Vladimirov (2010), Vladimirov (2011). There are four major incentives for the research described in this paper:
(1) To derive the averaged equations of non-degenerated oscillatory flows. Such flows appear in various applications and their study represents a necessary step towards the

systematic studies of various classes of degenerated flows.
(2) To make all the calculations and derivations elementary and free of any physical and mathematical assumptions (except the most common ones, like the existence of a differentiable solutions). To use only Eulerian description and Eulerian averaging operation, since they are the most transparent and allows to avoid additional steps and difficulties. For example, Eulerian description does not experience well-known difficulties with chaotic trajectories. Such a deliberately simple framework would allow to build further physical models based on a solid ground.
(3) To study the reduction of the Reynolds-stress-type terms to the drift-type terms, see Craik & Leibovich (1976), Riley (2001), Vladimirov (2010), Vladimirov (2011), Ilin & Morgulis (2011). This rather surprising reduction was discovered by Craik & Leibovich (1976) in the small-amplitude theory of Langmuir circulations, but its generality and importance in many other areas of fluid dynamics has not yet been recognized. In particular, it is interesting to find what else (besides the drift velocities) can be produced by the Reynolds-stresses-type terms.
(4) To understand how universal is the structure of the averaged equations. In order to describe universal features and variations of transformations of the Reynolds-stresses-type terms (or cross-correlations) we study three different problems: (i) a passive scalar admixture, (ii) a passive vectorial admixture (magnetic field in kinematic MHD), and (iii) an active vectorial admixture (vortex dynamics). This target is methodologically important: after establishing the universal building blocks of the theory one can use them in various degenerated cases; examples of such use can be found in Vladimirov (2010), Vladimirov (2011).

For the averaging and transforming of the governing equations we employ the two-timing method, see *e.g.* Nayfeh (1973), Kevorkian & Cole (1996). Our version of the method represents an elementary, systematic, and justifiable procedure introduced by Vladimirov (2005), Yudovich (2006), Vladimirov (2008), Vladimirov (2010). This procedure is complemented by consideration of the distinguished limits, that develops results of Vladimirov (2010), Vladimirov (2011). We also actively use the properties of commutators, which allow us to simplify analytical calculations. The employment of these methods and tools produces recurrent equations for both the averaged and oscillating parts of unknown fields. In all problems (i)-(iii) we present the calculations of three successive approximations (zeroth, first, and second).

In *Sect.*2 the notations are briefly introduced and the list the main definitions (of the averaging operation, *etc.*) is presented.

In *Sects.*3-5 we show that in all problems (i)-(iii) the averaged equations exhibit the universal structure: the zeroth approximations repeat the structure of the original equations; in the first and second approximation the Reynolds-stress-type terms are transformed into drift velocities and pseudo-diffusion, which are the same for all three problems; two additional terms (reminiscent of the *mean-field terms* in turbulence, see Moffatt (1978), Moffatt (1983)) also appear for magnetic fields and vortex dynamics. In particular, the averaged motion of a passive scalar admixture is described only by a drift in the first approximation, and by a combination of a drift and pseudo-diffusion in the second approximation. It is remarkable that all mean-field terms (including pseudo-diffusion) are expressed by invariant operators (Lie-derivatives) that measure the deviation of some tensors from their 'frozen-in' values.

*Sect.*6 is devoted to the notion of a distinguished limit. Here we explain why for the non-degenerated velocity fields a slow time-variable s coincides with physical time t. Such a coincidence allows us to build a theory using only two time-variables $s = t$ and $\tau = \sigma t$.

*Sect.*7 expands the area of applicability of all previously exposed results. We show

that for all three considered problems the high-frequency asymptotic solutions and the small-amplitude asymptotic solutions can be mathematically identical. This property is valid for s-independent velocity fields.

*Sect.*8 contains physical analysis of the nature of pseudo-diffusion. Our consideration suggests that this nature is purely kinematic, it is not related to physical diffusion.

*Sect.*9 (*Discussion*) is devoted to the restrictions and limitations of the introduced transformations of the governing equations, as well as to their possible generalizations.

2. Used Functions and Notations

The variables $\boldsymbol{x} = (x_1, x_2, x_3)$, t, s, and τ in the text below serve as dimensionless cartesian coordinates, physical time, slow time, and fast time. The used definitions, notations, and properties are itemized with bullets (•):

• The class $\mathbb{H}$ of *hat-functions* is defined as

$$\widehat{f} \in \mathbb{H}: \quad \widehat{f}(\boldsymbol{x}, s, \tau) = \widehat{f}(\boldsymbol{x}, s, \tau + 2\pi) \tag{2.1}$$

where the τ-dependence is always 2π-periodic; the dependencies on $\boldsymbol{x}$ and s are not specified.

• The subscripts t, τ, and s denote the related partial derivatives.

• For an arbitrary $\widehat{f} \in \mathbb{H}$ the *averaging operation* is

$$\langle \widehat{f}\, \rangle \equiv \frac{1}{2\pi} \int_{\tau_0}^{\tau_0 + 2\pi} \widehat{f}(\boldsymbol{x}, s, \tau)\, d\tau, \qquad \forall\ \tau_0 \tag{2.2}$$

where during the τ-integration $s = \text{const}$ and $\langle \widehat{f} \rangle$ does not depend on τ_0.

• The class $\mathbb{T}$ of *tilde-functions* is such that

$$\widetilde{f} \in \mathbb{T}: \quad \widetilde{f}(\boldsymbol{x}, s, \tau) = \widetilde{f}(\boldsymbol{x}, s, \tau + 2\pi), \quad \text{with} \quad \langle \widetilde{f} \rangle = 0, \tag{2.3}$$

Tilde-functions are also called purely oscillating functions; they represent a special case of hat-functions with the zero average.

• The class $\mathbb{B}$ of *bar-functions* is defined as

$$\overline{f} \in \mathbb{B}: \quad \overline{f}_\tau \equiv 0, \quad \overline{f}(\boldsymbol{x}, s) = \langle \overline{f}(\boldsymbol{x}, s) \rangle \tag{2.4}$$

• Any $\mathbb{H}$-function can be uniquely separated into its bar- and tilde- parts with the use of (2.2):

$$\widehat{f} = \overline{f} + \widetilde{f} \tag{2.5}$$

• *The $\mathbb{T}$-integration (or the tilde-integration)*: for a given $\widetilde{f}$ we introduce a new function $\widetilde{f}^\tau$ called the $\mathbb{T}$-integral of $\widetilde{f}$:

$$\widetilde{f}^\tau \equiv \int_0^\tau \widetilde{f}(\boldsymbol{x}, s, \rho)\, d\rho - \frac{1}{2\pi} \int_0^{2\pi} \Big(\int_0^\mu \widetilde{f}(\boldsymbol{x}, s, \rho)\, d\rho \Big)\, d\mu \tag{2.6}$$

• The unique solution of a PDE inside the tilde-class

$$\widetilde{f}_\tau \equiv \partial \widetilde{f} / \partial \tau = 0 \quad \Rightarrow \quad \widetilde{f} \equiv 0 \tag{2.7}$$

follows from (2.6).

• The τ-derivative of a tilde-function always represents a tilde-function. However the τ-integration of a tilde-function can produce a hat-function. An example: let us take $\widetilde{\phi} = \overline{\phi}_0 \sin \tau$ where $\overline{\phi}_0$ be an arbitrary bar-function: one can see that $\langle \widetilde{\phi}\, \rangle \equiv 0$, however

$\langle\int_0^\tau \widetilde{\phi}(\boldsymbol{x},s,\rho)d\rho\rangle = \overline{\phi}_0 \neq 0$, unless $\overline{\phi}_0 \equiv 0$. Formula (2.6) keeps the result of integration inside the $\mathbb{T}$-class.

• The $\mathbb{T}$-integration is inverse to the τ-differentiation $(\widetilde{f}^\tau)_\tau = (\widetilde{f}_\tau)^\tau = \widetilde{f}$; the proof is omitted.

• The product of two tilde-functions $\widetilde{f}$ and $\widetilde{g}$ represents a hat-function: $\widetilde{f}\widetilde{g} \equiv \widehat{F}$, say. Separating its tilde-part we write

$$\widetilde{F} = \widehat{F} - \langle\widehat{F}\rangle = \widetilde{f}\widetilde{g} - \langle\widetilde{f}\widetilde{g}\rangle = \{\widetilde{f}\widetilde{g}\} \tag{2.8}$$

where the notation $\{\cdot\}$ is introduced to avoid two levels of tildes.

• A dimensionless function $f = f(\boldsymbol{x},s,\tau)$ belongs to the class $\mathbb{O}(1)$

$$f \in \mathbb{O}(1) \tag{2.9}$$

if $f = O(1)$ and all partial $\boldsymbol{x}$-, s-, and τ-derivatives of f (required for our consideration), are also $O(1)$.

• All large or small parameters in this paper are represented by various degrees of a large parameter σ (which represents dimensionless frequency) only; large or small parameters appear as explicit multipliers in all formulae, while all functions always belong to $\mathbb{O}(1)$-class.

• The commutator of two vector-fields $\boldsymbol{f}$ and $\boldsymbol{g}$ is

$$[\boldsymbol{f},\boldsymbol{g}] \equiv (\boldsymbol{g}\cdot\nabla)\boldsymbol{f} - (\boldsymbol{f}\cdot\nabla)\boldsymbol{g}, \tag{2.10}$$

It is antisymmetric and satisfies Jacobi's identity for any vector-fields $\boldsymbol{f}$, $\boldsymbol{g}$, and $\boldsymbol{h}$:

$$[\boldsymbol{f},\boldsymbol{g}] = -[\boldsymbol{g},\boldsymbol{f}], \quad [\boldsymbol{f},[\boldsymbol{g},\boldsymbol{h}]] + [\boldsymbol{h},[\boldsymbol{f},\boldsymbol{g}]] + [\boldsymbol{g},[\boldsymbol{h},\boldsymbol{f}]] = 0 \tag{2.11}$$

• From (2.5) and (2.4) one can see that

$$\widehat{f}_\tau = \widetilde{f}_\tau, \quad \langle\widehat{f}_\tau\rangle = \langle\widetilde{f}_\tau\rangle = 0 \tag{2.12}$$

• As the average operation (2.2) is proportional to the integration over τ, the integration by parts yields

$$\langle\widetilde{f}\widetilde{g}_\tau\rangle = \langle(\widetilde{f}\widetilde{g})_\tau\rangle - \langle\widetilde{f}_\tau\widetilde{g}\rangle = -\langle\widetilde{f}_\tau\widetilde{g}\rangle = -\langle\widetilde{f}_\tau\widehat{g}\rangle \tag{2.13}$$

$$\langle\widetilde{f}_\tau\widetilde{g}\widetilde{h}\rangle + \langle\widetilde{f}\widetilde{g}_\tau\widetilde{h}\rangle + \langle\widetilde{f}\widetilde{g}\widetilde{h}_\tau\rangle = 0 \tag{2.14}$$

$$\langle\widetilde{f}\widetilde{g}^\tau\rangle = \langle(\widetilde{f}^\tau\widetilde{g}^\tau)_\tau\rangle - \langle\widetilde{f}^\tau\widetilde{g}\rangle = -\langle\widetilde{f}^\tau\widetilde{g}\rangle = -\langle\widetilde{f}^\tau\widehat{g}\rangle \tag{2.15}$$

$$\langle[\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{g}}_\tau]\rangle = -\langle[\widetilde{\boldsymbol{f}}_\tau,\widetilde{\boldsymbol{g}}]\rangle = -\langle[\widetilde{\boldsymbol{f}}_\tau,\widehat{\boldsymbol{g}}]\rangle,\ \langle[\widetilde{\boldsymbol{f}},\widetilde{\boldsymbol{g}}^\tau]\rangle = -\langle[\widetilde{\boldsymbol{f}}^\tau,\widetilde{\boldsymbol{g}}]\rangle = -\langle[\widetilde{\boldsymbol{f}}^\tau,\widehat{\boldsymbol{g}}]\rangle \tag{2.16}$$

Many of the above definitions and terms are different from the ones used in various branches of physics and fluid dynamics. We have introduced our own terms in order to avoid ambiguities.

3. Scalar Admixture

3.1. *Two-timing formulation*

The equation for a passive scalar admixture c transported by a given velocity field $\boldsymbol{u}^*$ in an incompressible fluid is

$$\frac{\partial c}{\partial t^*} + \boldsymbol{u}^*\cdot\nabla^* c = 0, \quad \nabla^*\cdot\boldsymbol{u}^* = 0 \tag{3.1}$$

where asterisks mark dimensional variables, t^*-time, $\boldsymbol{x}^* = (x_1^*, x_2^*, x_3^*)$-cartesian coordinates, $\nabla^* = (\partial/\partial x_1^*, \partial/\partial x_2^*, \partial/\partial x_3^*)$. This paper id focused on the transformations of

the governing equations, therefore the geometry a flow domain and particular boundary conditions can be introduced at later stages of the research (see Discussion). We accept that the considered class of oscillatory flows $\boldsymbol{u}^* = \boldsymbol{u}^*(\boldsymbol{x}^*, t^*; \sigma^*)$ possesses independent characteristic scales of velocity U, length L, time T, and frequency σ^*

$$U, \quad L, \quad T, \quad \sigma^*; \quad T_i \equiv L/U, \quad T = T_i \tag{3.2}$$

where T_i is a dependent (intrinsic) time-scale. It will be shown by the distinguished limit consideration that for non-degenerated flows the scales T and T_i are of similar order (one can accept that $T = kT_i$ with a constant $k \sim 1$; however without any restriction of generality this constant can be taken as $k = 1$). Then dimensionless (not asteriated) variables and frequency are

$$\boldsymbol{x} \equiv \boldsymbol{x}^*/L, \quad t^* \equiv t/T, \quad \boldsymbol{u} \equiv \boldsymbol{u}^*/U, \quad \sigma \equiv \sigma^* T; \quad 1/\sigma \ll 1 \tag{3.3}$$

where $1/\sigma$ is our basic small parameter. Due to the linearity of (3.1) we consider the scalar field c being dimensionless. The dimensionless version of (3.1) is

$$c_t + \boldsymbol{u} \cdot \nabla c = 0 \tag{3.4}$$

The *given* oscillating velocity $\boldsymbol{u}$ is taken as a hat-function (2.1)

$$\boldsymbol{u} = \widehat{\boldsymbol{u}}(\boldsymbol{x}, t, \sigma t) \quad \text{or} \quad \boldsymbol{u} = \widehat{\boldsymbol{u}}(\boldsymbol{x}, s, \tau) \quad \text{with } \tau = \sigma t, \ s = t \tag{3.5}$$

where s and τ are two *mutually dependent* time-variables (s is *slow time* and τ is *fast time*). We assume that the solution of (3.4),(3.5) also represents a hat-function

$$c = \widehat{c}(\boldsymbol{x}, s, \tau) \tag{3.6}$$

In order to justify this assumption one needs to build such an unique solution for arbitrary initial data. The chain rule transforms (3.4) to

$$\left(\frac{\partial}{\partial \tau} + \frac{1}{\sigma}\frac{\partial}{\partial s}\right)\widehat{c} + \frac{1}{\sigma}\widehat{\boldsymbol{u}} \cdot \nabla \widehat{c} = 0 \tag{3.7}$$

Eqn.(3.7) contains the only small parameter

$$\varepsilon \equiv \frac{1}{T\sigma^*} = \frac{1}{\sigma} \tag{3.8}$$

Here one must make an auxiliary (but technically essential) step: after the use of the chain rule (3.7) the variables s and τ are considered to be (temporarily) *mutually independent*:

$$\tau \quad \text{and} \quad s \quad \text{are independent variables (temporarily)} \tag{3.9}$$

From the mathematical viewpoint the rise of the number of independent variables in a PDE represents a very radical step, which leads to an entirely new PDE. This step has to be justified *a posteriori* by showing that the error of the obtained solution (rewritten back to the original variable t and substituted into the original equation (3.4)), is small.

Hence there is the equation which is going to be considered in detail

$$\widehat{c}_\tau + \varepsilon(\widehat{\boldsymbol{u}} \cdot \nabla)\widehat{c} + \varepsilon \widehat{c}_s = 0, \quad \operatorname{div} \widehat{\boldsymbol{u}} = 0 \quad \varepsilon \equiv 1/\sigma \to 0 \tag{3.10}$$

3.2. *Results for the successive approximations of scalar field*

We look for the solutions of (3.10) in the form of regular series

$$(\widehat{c}, \widehat{\boldsymbol{u}}) = \sum_{k=0}^{\infty} \varepsilon^k (\widehat{c}_k, \widehat{\boldsymbol{u}}_k); \quad \widehat{c}_k, \widehat{\boldsymbol{u}}_k \in \mathbb{H} \cap \mathbb{O}(1), \quad k = 0, 1, 2, \dots \tag{3.11}$$

where all $\widehat{\boldsymbol{u}}_k$ are given. The main approximation of the general velocity field $\widehat{\boldsymbol{u}}$ is not degenerated:

$$\widehat{\boldsymbol{u}}_0 = \overline{\boldsymbol{u}}_0 + \widetilde{\boldsymbol{u}}_0, \quad \text{with} \quad \overline{\boldsymbol{u}}_0 \not\equiv 0 \quad \text{and} \quad \widetilde{\boldsymbol{u}}_0 \not\equiv 0 \tag{3.12}$$

The substitution of (3.11) into (3.10) produces the equations for the first three successive approximations

$$\widehat{c}_{0\tau} = 0 \tag{3.13}$$
$$\widehat{c}_{1\tau} + \widehat{\boldsymbol{u}}_0 \cdot \nabla \widehat{c}_0 + \widehat{c}_{0s} = 0 \tag{3.14}$$
$$\widehat{c}_{2\tau} + \widehat{\boldsymbol{u}}_1 \cdot \nabla \widehat{c}_0 + \widehat{\boldsymbol{u}}_0 \cdot \nabla \widehat{c}_1 + \widehat{c}_{1s} = 0 \tag{3.15}$$
$$\widehat{c}_{3\tau} + \widehat{\boldsymbol{u}}_2 \cdot \nabla \widehat{c}_0 + \widehat{\boldsymbol{u}}_0 \cdot \nabla \widehat{c}_2 + \widehat{\boldsymbol{u}}_1 \cdot \nabla \widehat{c}_1 + \widehat{c}_{2s} = 0 \tag{3.16}$$
$$\widehat{c}_k = \overline{c}_k(\boldsymbol{x}, t) + \widetilde{c}_k(\boldsymbol{x}, t, \tau), \ \overline{c}_k \in \mathbb{B} \cap \mathbb{O}(1), \ \widetilde{c}_k \in \mathbb{T} \cap \mathbb{O}(1), \ k = 0, 1, 2, 3$$

The detailed solution/transformation of (3.13)-(3.16) is given in *Appendix A*. Here we formulate the result. The truncated general solution $\widehat{c}^{[3]}$ is

$$\widehat{c}^{[3]} = \widehat{c}_0 + \varepsilon \widehat{c}_1 + \varepsilon^2 \widehat{c}_2 + \varepsilon^3 \widehat{c}_3 \tag{3.17}$$

Its bar-parts $\overline{c}_k$ satisfy the equations

$$\overline{c}_{0s} + \overline{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_0 = 0 \tag{3.18}$$
$$\overline{c}_{1s} + \overline{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_1 + (\overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0) \cdot \nabla \overline{c}_0 = 0 \tag{3.19}$$
$$\overline{c}_{2s} + \overline{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_2 + (\overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0) \cdot \nabla \overline{c}_1 + \tag{3.20}$$
$$(\overline{\boldsymbol{u}}_2 + \overline{\boldsymbol{V}}_{10} + \overline{\boldsymbol{V}}_1 + \overline{\boldsymbol{V}}_\chi) \cdot \nabla \overline{c}_1 = \frac{\partial}{\partial x_i}\left(\overline{\chi}_{ik} \frac{\partial \overline{c}_0}{\partial x_k}\right)$$
$$\overline{\boldsymbol{V}}_0 \equiv \frac{1}{2}\langle [\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{\xi}}]\rangle, \quad \overline{\boldsymbol{V}}_1 \equiv \frac{1}{3}\langle [[\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{\xi}}], \widetilde{\boldsymbol{\xi}}]\rangle, \quad \widetilde{\boldsymbol{\xi}} \equiv \widetilde{\boldsymbol{u}}_0^\tau \tag{3.21}$$
$$\overline{\boldsymbol{V}}_{10} \equiv \langle [\widetilde{\boldsymbol{u}}_1, \widetilde{\boldsymbol{\xi}}]\rangle, \quad \boldsymbol{V}_\chi \equiv \frac{1}{2}\langle [\widetilde{\boldsymbol{\xi}}, \mathfrak{L}\widetilde{\boldsymbol{\xi}}]\rangle, \quad \overline{\chi}_{ik} \equiv \frac{1}{2}\mathfrak{L}\langle \widetilde{\xi}_i \widetilde{\xi}_k\rangle, \tag{3.22}$$

where the operator $\mathfrak{L}$ (Lie-derivative) is such that its action on $\widetilde{\boldsymbol{\xi}}$ is defined as

$$\mathfrak{L}\widetilde{\boldsymbol{\xi}} \equiv \widetilde{\boldsymbol{\xi}}_s + [\widetilde{\boldsymbol{\xi}}, \overline{\boldsymbol{u}}_0] \tag{3.23}$$

and its action on any (constructed from $\widetilde{\boldsymbol{\xi}}$) tensorial field $f_{ik\ldots}$ is such that $\mathfrak{L}\widetilde{\boldsymbol{\xi}} = 0$ implies $\mathfrak{L}f_{ik\ldots} = 0$. In particular,

$$\mathfrak{L}\langle \widetilde{\xi}_i \widetilde{\xi}_k\rangle \equiv (\partial_s + \overline{\boldsymbol{u}}_0 \cdot \nabla)\langle \widetilde{\xi}_i \widetilde{\xi}_k\rangle - \frac{\partial \overline{u}_{0k}}{\partial x_m}\langle \widetilde{\xi}_i \widetilde{\xi}_m\rangle - \frac{\partial \overline{u}_{0i}}{\partial x_m}\langle \widetilde{\xi}_k \widetilde{\xi}_m\rangle \tag{3.24}$$

Formulae (3.23) and (3.24) are also known as 'frozen-in' operators for a vectorial field and tensorial field. After (3.18)-(3.20) are solved, the tilde-parts $\widetilde{c}_k$ of (4.11) are given by the recurrent expressions

$$\widetilde{c}_0 \equiv 0, \tag{3.25}$$
$$\widetilde{c}_1 = -\widetilde{\boldsymbol{\xi}} \cdot \nabla \overline{c}_0, \tag{3.26}$$
$$\widetilde{c}_2 = -\widetilde{\boldsymbol{u}}_1^\tau \cdot \nabla \overline{c}_0 - \overline{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_1^\tau - \widetilde{\boldsymbol{\xi}} \cdot \nabla \overline{c}_1 - \{\widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_1\}^\tau - \widetilde{c}_{1s}^\tau, \tag{3.27}$$
$$\widetilde{c}_{3\tau} = -\widetilde{\boldsymbol{u}}_2^\tau \cdot \nabla \overline{c}_0 - \widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla \overline{c}_2 - \overline{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_2^\tau - \widetilde{\boldsymbol{u}}_1^\tau \cdot \nabla \overline{c}_1 - \overline{\boldsymbol{u}}_1 \cdot \nabla \widetilde{c}_1^\tau - \tag{3.28}$$
$$\{\widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_2\}^\tau - \{\widetilde{\boldsymbol{u}}_1 \cdot \nabla \widetilde{c}_1\}^\tau - \widetilde{c}_{2s}^\tau$$

Three averaged equations (3.18)-(3.20) can be written as a single advection-pseudo-

diffusion equation (with the error $O(\varepsilon^3)$)

$$(\partial_s + \overline{\boldsymbol{v}} \cdot \nabla)\,\overline{c} = \frac{\partial}{\partial x_i}\left(\overline{\kappa}_{ik}\frac{\partial \overline{c}}{\partial x_k}\right) \tag{3.29}$$

$$\overline{c} = \overline{c}^{[2]} = \overline{c}_0 + \varepsilon\overline{c}_1 + \varepsilon^2\overline{c}_2, \quad \overline{\boldsymbol{v}} = \overline{\boldsymbol{u}}^{[2]} + \overline{\boldsymbol{V}}^{[2]} \tag{3.30}$$

$$\overline{\boldsymbol{u}}^{[2]} = \overline{\boldsymbol{u}}_0 + \varepsilon\overline{\boldsymbol{u}}_1 + \varepsilon^2\overline{\boldsymbol{u}}_2, \quad \overline{\boldsymbol{V}}^{[2]} = \varepsilon\overline{\boldsymbol{V}}_0 + \varepsilon^2(\overline{\boldsymbol{V}}_{10} + \overline{\boldsymbol{V}}_1 + \overline{\boldsymbol{V}}_\chi) \tag{3.31}$$

$$\overline{\kappa}_{ik} = \overline{\chi}^{[2]}_{ik} = \varepsilon^2\overline{\chi}_{ik} \tag{3.32}$$

where all the coefficients are given in (3.21),(3.22). Eqn. (3.29) shows that (with a given precision $O(\varepsilon^2)$) the motion of $\overline{c}$ represents an advection with velocity $\overline{\boldsymbol{v}}$ and pseudo-diffusion with matrix coefficient $\overline{\kappa}_{ik} = \varepsilon^2\overline{\chi}_{ik}$.

One can write the truncated solution

$$\widehat{c}^{[3]} = \overline{c}_0 + \overline{c}_1 + \widetilde{c}_1 + \overline{c}_2 + \widetilde{c}_2 + \overline{c}_3 + \widetilde{c}_3 \tag{3.33}$$

where all the terms except $\overline{c}_3$ are determined by the above equations. After rewriting (3.33) back to the original variable t ($s = t, \tau = \sigma t$) and its substitution into (3.4) one can show that the error term in the RHS of the equation is $O(\varepsilon^3)$. This estimation might be considered as *mathematical justification* of the used procedure, including its most sensitive part (3.9).

4. Kinematic MHD-equations

4.1. *Two-timing formulation of the kinematic MHD*

The equation for a magnetic field $\boldsymbol{h}$ to be 'frozen' into a given oscillating incompressible velocity field $\boldsymbol{u}^*(\boldsymbol{x}^*, t^*; \sigma^*)$ is

$$\frac{\partial \boldsymbol{h}}{\partial t^*} + [\boldsymbol{h}, \boldsymbol{u}^*]^* = 0, \quad \nabla^* \cdot \boldsymbol{u}^* = 0, \quad \nabla^* \cdot \boldsymbol{h} = 0 \tag{4.1}$$

where all the notations are the same as in *Sect.*3.1; $[\,\cdot\,,\cdot\,]^*$ stands for dimensional commutator (2.10), and $\boldsymbol{h}$ is taken dimensionless due to linearity. The analysis of dimensions is the same as in *Sect.*3.1; the dimensionless version of (4.1) is

$$\boldsymbol{h}_t + [\boldsymbol{h}, \boldsymbol{u}] = 0, \quad \nabla \cdot \boldsymbol{u} = 0, \quad \nabla \cdot \boldsymbol{h} = 0 \tag{4.2}$$

The *given* oscillating velocity is taken as a hat-function (2.1)

$$\boldsymbol{u} = \widehat{\boldsymbol{u}}(\boldsymbol{x}, s, \tau); \quad \text{with } \tau \equiv \sigma t,\ s \equiv t \tag{4.3}$$

and the solution of (4.2),(4.3) is also taken as a hat-function

$$\boldsymbol{h} = \widehat{\boldsymbol{h}}(\boldsymbol{x}, s, \tau) \tag{4.4}$$

Then the following equation has to be studied

$$\widehat{\boldsymbol{h}}_\tau + \varepsilon[\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{u}}] + \varepsilon\widehat{\boldsymbol{h}}_s = 0, \quad \varepsilon \to 0 \tag{4.5}$$

where the variables τ and s are again temporarily independent (3.9).

4.2. *Results for the successive approximations of a magnetic field*

We look for the solution of (4.5) in the form of regular series

$$(\widehat{\boldsymbol{h}}, \widehat{\boldsymbol{u}}) = \sum_{k=0}^{\infty} \varepsilon^k(\widehat{\boldsymbol{h}}_k, \widehat{\boldsymbol{u}}_k); \quad \widehat{\boldsymbol{h}}_k, \widehat{\boldsymbol{u}}_k \in \mathbb{H} \cap \mathbb{O}(1), \quad k = 0, 1, 2, \dots \tag{4.6}$$

where all $\widehat{\boldsymbol{u}}_k$ are given, and (as in *Sect.*3.2) we consider a prescribed non-degenerated velocity field: $\overline{\boldsymbol{u}}_0 \not\equiv 0$ and $\widetilde{\boldsymbol{u}}_0 \not\equiv 0$. The substitution of (4.6) into (4.5) produces the equations for the first three successive approximations

$$\widehat{\boldsymbol{h}}_{0\tau} = 0 \tag{4.7}$$

$$\widehat{\boldsymbol{h}}_{1\tau} + [\widehat{\boldsymbol{h}}_0, \widehat{\boldsymbol{u}}_0] + \widehat{\boldsymbol{h}}_{0s} = 0 \tag{4.8}$$

$$\widehat{\boldsymbol{h}}_{2\tau} + \widehat{\boldsymbol{h}}_{1s} + [\widehat{\boldsymbol{h}}_0, \widehat{\boldsymbol{u}}_1] + [\widehat{\boldsymbol{h}}_1, \widehat{\boldsymbol{u}}_0] = 0 \tag{4.9}$$

$$\widehat{\boldsymbol{h}}_{3\tau} + \widehat{\boldsymbol{h}}_{2s} + [\widehat{\boldsymbol{h}}_0, \widehat{\boldsymbol{u}}_2] + [\widehat{\boldsymbol{h}}_2, \widehat{\boldsymbol{u}}_0] + [\widehat{\boldsymbol{h}}_1, \widehat{\boldsymbol{u}}_1] = 0 \tag{4.10}$$

$$\widehat{\boldsymbol{h}}_k = \overline{\boldsymbol{h}}_k(\boldsymbol{x}, t) + \widetilde{\boldsymbol{h}}_k(\boldsymbol{x}, t, \tau),\ \overline{\boldsymbol{h}}_k \in \mathbb{B} \cap \mathbb{O}(1),\ \widetilde{\boldsymbol{h}}_k \in \mathbb{T} \cap \mathbb{O}(1),\ k = 0, 1, 2, 3$$

The detailed solution/transformation of (4.7)-(4.10) is given in *Appendix B*. Here we formulate the result. The truncated general solution $\widehat{\boldsymbol{h}}^{[3]}$ is

$$\widehat{\boldsymbol{h}}^{[3]} = \widehat{\boldsymbol{h}}_0 + \varepsilon \widehat{\boldsymbol{h}}_1 + \varepsilon^2 \widehat{\boldsymbol{h}}_2 + \varepsilon^3 \widehat{\boldsymbol{h}}_3 \tag{4.11}$$

The bar-parts $\overline{\boldsymbol{h}}_k$ satisfy the equations

$$\overline{\boldsymbol{h}}_{0s} + [\overline{\boldsymbol{h}}_0, \overline{\boldsymbol{u}}_0] = 0 \tag{4.12}$$

$$\overline{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_0, \overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0] + [\overline{\boldsymbol{h}}_1, \overline{\boldsymbol{u}}_0] = 0 \tag{4.13}$$

$$\overline{\boldsymbol{h}}_{2s} + [\overline{\boldsymbol{h}}_2, \overline{\boldsymbol{u}}_0] + [\overline{\boldsymbol{h}}_1, (\overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0)] + [\overline{\boldsymbol{h}}_0, (\overline{\boldsymbol{u}}_2 + \overline{\boldsymbol{V}}_{10} + \overline{\boldsymbol{V}}_1 + \overline{\boldsymbol{V}}_\chi)] = \frac{\partial}{\partial x_i}\left(\overline{\chi}_{ik} \frac{\partial \overline{h}_{0i}}{\partial x_k}\right) - \overline{b}_{ikl} \frac{\partial \overline{h}_{0l}}{\partial x_k} - \overline{a}_{ik} \overline{h}_k \tag{4.14}$$

were the notations are the same as in (3.21),(3.22) and

$$\overline{b}_{ikl} \equiv \mathfrak{L}\left\langle \widetilde{\xi}_k \frac{\partial \widetilde{\xi}_i}{\partial x_l} \right\rangle, \quad \overline{a}_{ik} \equiv \mathfrak{L}\left\langle \widetilde{\xi}_l \frac{\partial^2 \widetilde{\xi}_i}{\partial x_l \partial x_k} - \frac{\partial \widetilde{\xi}_l}{\partial x_k} \frac{\partial \widetilde{\xi}_i}{\partial x_l} \right\rangle \tag{4.15}$$

The operator $\mathfrak{L}$ is Lie-derivative, see (3.23),(3.24). After (4.12)-(4.14) are solved, the tilde-parts $\widetilde{\boldsymbol{h}}_k$ of (4.11) are given by the recurrent expressions

$$\widetilde{\boldsymbol{h}}_0 \equiv 0, \tag{4.16}$$

$$\widetilde{\boldsymbol{h}}_1 = [\widetilde{\boldsymbol{u}}_0^\tau, \overline{\boldsymbol{h}}_0] \tag{4.17}$$

$$\widetilde{\boldsymbol{h}}_2 = -\widetilde{\boldsymbol{h}}_{1s}^\tau - [\overline{\boldsymbol{h}}_0, \widetilde{\boldsymbol{u}}_1^\tau] - [\widetilde{\boldsymbol{h}}_1^\tau, \overline{\boldsymbol{u}}_0] - [\overline{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0^\tau] - \{[\widetilde{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0]\}^\tau \tag{4.18}$$

$$\widetilde{\boldsymbol{h}}_3 = -\widetilde{\boldsymbol{h}}_{2s}^\tau - [\overline{\boldsymbol{h}}_0, \widetilde{\boldsymbol{u}}_2^\tau] - [\widetilde{\boldsymbol{h}}_2^\tau, \overline{\boldsymbol{u}}_0] - [\overline{\boldsymbol{h}}_2, \widetilde{\boldsymbol{u}}_0^\tau] - [\widetilde{\boldsymbol{h}}_1^\tau, \overline{\boldsymbol{u}}_1] - [\overline{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_1^\tau] - \{[\widetilde{\boldsymbol{h}}_2, \widetilde{\boldsymbol{u}}_0]\}^\tau - \{[\widetilde{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_1]\}^\tau \tag{4.19}$$

Three averaged equations (4.12)-(4.14) can be written as a single equation that combines advection, pseudo-diffusion, and other mean-field-type terms (with the error $O(\varepsilon^3)$)

$$\overline{\boldsymbol{h}}_s + [\overline{\boldsymbol{h}}, \overline{\boldsymbol{v}}] = \frac{\partial}{\partial x_k}\left(\overline{\kappa}_{kl} \frac{\partial \overline{h}_{0i}}{\partial x_l}\right) - \overline{B}_{ikl} \frac{\partial \overline{h}_{0l}}{\partial x_k} - \overline{A}_{ik} \overline{h}_k \tag{4.20}$$

$$\overline{\boldsymbol{h}} = \overline{\boldsymbol{h}}^{[2]} = \overline{\boldsymbol{h}}_0 + \varepsilon \overline{\boldsymbol{h}}_1 + \varepsilon^2 \overline{\boldsymbol{h}}_2 \tag{4.21}$$

The notation for $\overline{\boldsymbol{v}}$, $\overline{\boldsymbol{u}}^{[2]}$, $\overline{\boldsymbol{V}}^{[2]}$, and $\overline{\kappa}_{ik}$ are the same as in (3.30), while $\overline{B}_{ikl} \equiv \varepsilon^2 \overline{b}_{ikl}$ and $\overline{A}_{ik} \equiv \varepsilon^2 \overline{a}_{ik}$. Eqn. (4.20) shows that the evolution of $\overline{\boldsymbol{h}}$ represents (with the error $O(\varepsilon^3)$) an advection and stretching by velocity $\overline{\boldsymbol{v}}$, pseudo-diffusion with matrix coefficient $\overline{\kappa}_{ik} = \varepsilon^2 \overline{\chi}_{ik}$, and other mean-field deformations. The terms with $\overline{B}_{ikl}$ and $\overline{A}_{ik}$ are essentially new in comparison with (3.29). The construction and justification of the truncated solution can be performed in a similar to *Sect.*3 way.

5. Euler's equations

5.1. *Two-timing problem of vorticity dynamics*

The governing equation for dynamics of an inviscid incompressible fluid with velocity field $\boldsymbol{u}^*$ and vorticity $\boldsymbol{\omega}^*$ is taken in the vorticity form

$$\frac{\partial \boldsymbol{\omega}^*}{\partial t^*} + [\boldsymbol{\omega}^*, \boldsymbol{u}^*]^* = 0, \qquad \nabla^* \cdot \boldsymbol{u}^* = 0 \tag{5.1}$$

$$\boldsymbol{\omega}^* \equiv \nabla^* \times \boldsymbol{u}^* \tag{5.2}$$

where all the notations are the same as in *Sects.*3.1 and 4.1. The equations (5.1) are mathematically the same as (4.1) in MHD-kinematics. However, they are complemented by an additional constraint (5.2). As a result the velocity field represents an unknown function and can not be considered as the prescribed one. It is remarkable, that this very sound difference does not have any impact on all the results of *Sect.*4 and the derivations of *Appendix B*. The sufficient (for the derivation of the averaged equations for vorticity) operations are: we have to replace $\boldsymbol{h}$ by $\boldsymbol{\omega}$ and write the constraint (5.2) additionally to all equations. Of course, the same averaged equations can be obtained independently.

The system of dimensionless equations under consideration can be written as:

$$\widehat{\boldsymbol{\omega}}_\tau + \varepsilon[\widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{u}}] + \varepsilon\widehat{\boldsymbol{\omega}}_s = 0, \quad \widehat{\boldsymbol{\omega}} = \nabla \times \widehat{\boldsymbol{u}}, \quad \operatorname{div} \widehat{\boldsymbol{u}} = 0; \quad \varepsilon \to 0 \tag{5.3}$$

We are looking for the solution of these equations in the form of regular series

$$(\widehat{\boldsymbol{\omega}}, \widehat{\boldsymbol{u}}) = \sum_{k=0}^{\infty} \varepsilon^k (\widehat{\boldsymbol{\omega}}_k, \widehat{\boldsymbol{u}}_k), \quad \widehat{\boldsymbol{\omega}}_k \equiv \nabla \times \widehat{\boldsymbol{u}}_k; \quad \widehat{\boldsymbol{\omega}}_k, \widehat{\boldsymbol{u}}_k \in \mathbb{H} \cap \mathbb{O}(1),\ k = 0, 1, 2, \dots \tag{5.4}$$

where all $\widehat{\boldsymbol{u}}_k$ represent unknown functions. Here we formulate the result. The truncated general solution $\widehat{\boldsymbol{\omega}}^{[3]}$ is

$$\widehat{\boldsymbol{\omega}}^{[3]} = \widehat{\boldsymbol{\omega}}_0 + \varepsilon\widehat{\boldsymbol{\omega}}_1 + \varepsilon^2\widehat{\boldsymbol{\omega}}_2 + \varepsilon^3\widehat{\boldsymbol{\omega}}_3 \tag{5.5}$$

The bar-parts $\overline{\boldsymbol{\omega}}_k$ satisfy the equations

$$\overline{\boldsymbol{\omega}}_{0s} + [\overline{\boldsymbol{\omega}}_0, \overline{\boldsymbol{u}}_0] = 0 \tag{5.6}$$

$$\overline{\boldsymbol{\omega}}_{1s} + [\overline{\boldsymbol{\omega}}_0, \overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0] + [\overline{\boldsymbol{\omega}}_1, \overline{\boldsymbol{u}}_0] = 0 \tag{5.7}$$

$$\begin{aligned}&\overline{\boldsymbol{\omega}}_{2s} + [\overline{\boldsymbol{\omega}}_2, \overline{\boldsymbol{u}}_0] + [\overline{\boldsymbol{\omega}}_1, (\overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0)] + [\overline{\boldsymbol{\omega}}_0, (\overline{\boldsymbol{u}}_2 + \overline{\boldsymbol{V}}_{10} + \overline{\boldsymbol{V}}_1 + \overline{\boldsymbol{V}}_\chi)] = \\ &\frac{\partial}{\partial x_i}\left(\overline{\chi}_{ik}\frac{\partial \overline{\omega}_{0i}}{\partial x_k}\right) - \overline{b}_{ikl}\frac{\partial \overline{\omega}_{0l}}{\partial x_k} - \overline{a}_{ik}\overline{\omega}_k\end{aligned} \tag{5.8}$$

All these equations are complemented by the constraints $\overline{\boldsymbol{\omega}}_k \equiv \nabla \times \overline{\boldsymbol{u}}_k$ for $k = 0, 1, 2$. After (5.6)-(5.8) are solved, the tilde-parts $\widetilde{\boldsymbol{\omega}}_k$ of (4.11) are given by the recurrent expressions

$$\widetilde{\boldsymbol{\omega}}_0 \equiv 0, \tag{5.9}$$

$$\widetilde{\boldsymbol{\omega}}_1 = [\widetilde{\boldsymbol{u}}_0^\tau, \overline{\boldsymbol{\omega}}_0] \tag{5.10}$$

$$\widetilde{\boldsymbol{\omega}}_2 = -\widetilde{\boldsymbol{\omega}}_{1s}^\tau - [\overline{\boldsymbol{\omega}}_0, \widetilde{\boldsymbol{u}}_1^\tau] - [\widetilde{\boldsymbol{\omega}}_1^\tau, \overline{\boldsymbol{u}}_0] - [\overline{\boldsymbol{\omega}}_1, \widetilde{\boldsymbol{u}}_0^\tau] - \{[\widetilde{\boldsymbol{\omega}}_1, \widetilde{\boldsymbol{u}}_0]\}^\tau \tag{5.11}$$

$$\begin{aligned}\widetilde{\boldsymbol{\omega}}_3 = &-\widetilde{\boldsymbol{\omega}}_{2s}^\tau - [\overline{\boldsymbol{\omega}}_0, \widetilde{\boldsymbol{u}}_2^\tau] - [\widetilde{\boldsymbol{\omega}}_2^\tau, \overline{\boldsymbol{u}}_0] - [\overline{\boldsymbol{\omega}}_2, \widetilde{\boldsymbol{u}}_0^\tau] - \\ &[\widetilde{\boldsymbol{\omega}}_1^\tau, \overline{\boldsymbol{u}}_1] - [\overline{\boldsymbol{\omega}}_1, \widetilde{\boldsymbol{u}}_1^\tau] - \{[\widetilde{\boldsymbol{\omega}}_2, \widetilde{\boldsymbol{u}}_0]\}^\tau - \{[\widetilde{\boldsymbol{\omega}}_1, \widetilde{\boldsymbol{u}}_1]\}^\tau\end{aligned} \tag{5.12}$$

All these equations are complemented by the constraints $\widetilde{\boldsymbol{\omega}}_k \equiv \nabla \times \widetilde{\boldsymbol{u}}_k$ for $k = 0, 1, 2$. Three averaged equations (5.6)-(5.8) can be written as a single equation that combines

advection, pseudo-diffusion, and other mean-field-type terms (with the error $O(\varepsilon^3)$)

$$\overline{\boldsymbol{\omega}}_s + [\overline{\boldsymbol{\omega}}, \overline{\boldsymbol{v}}] = \frac{\partial}{\partial x_k}\left(\overline{\chi}_{kl}\frac{\partial \overline{\omega}_{0i}}{\partial x_l}\right) - \overline{b}_{ikl}\frac{\partial \overline{\omega}_{0l}}{\partial x_k} - \overline{a}_{ik}\overline{\omega}_k \tag{5.13}$$

$$\overline{\boldsymbol{v}} \equiv \overline{\boldsymbol{u}}_0 + \varepsilon(\overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0) + \varepsilon^2(\overline{\boldsymbol{u}}_2 + \overline{\boldsymbol{V}}_{10} + \overline{\boldsymbol{V}}_1 + \overline{\boldsymbol{V}}_\chi) \tag{5.14}$$

$$\overline{\kappa}_{ik} = \overline{\chi}^{[2]}_{ik} = \varepsilon^2\overline{\chi}_{ik} \tag{5.15}$$

$$\overline{\boldsymbol{\omega}} = \overline{\boldsymbol{\omega}}^{[2]} = \overline{\boldsymbol{\omega}}_0 + \varepsilon\overline{\boldsymbol{\omega}}_1 + \varepsilon^2\overline{\boldsymbol{\omega}}_2 \tag{5.16}$$

Eqn. (5.13) shows that the evolution of $\overline{\boldsymbol{\omega}}$ represents (with the error $O(\varepsilon^3)$) an advection and stretching by velocity $\overline{\boldsymbol{v}}$, pseudo-diffusion with matrix coefficient $\overline{\kappa}_{ik} = \varepsilon^2\overline{\chi}_{ik}$, and other mean-field deformations. It is *very* important that $\overline{\boldsymbol{\omega}} \neq \nabla \times \overline{\boldsymbol{v}}$.

One can now write the truncated solution

$$\widehat{\boldsymbol{\omega}}^{[3]} = \overline{\boldsymbol{\omega}}_0 + \overline{\boldsymbol{\omega}}_1 + \widetilde{\boldsymbol{\omega}}_1 + \overline{\boldsymbol{\omega}}_2 + \widetilde{\boldsymbol{\omega}}_2 + \overline{\boldsymbol{\omega}}_3 + \widetilde{\boldsymbol{\omega}}_3 \tag{5.17}$$

where all the terms except $\overline{\boldsymbol{\omega}}_3$ are determined by the above equations. Its mathematical justification is similar to the one presented in *Sect.*3.

The difference between the results for vortex dynamics and the results for a magnetic field is essential: in the former case velocity is prescribed, while (5.6)-(5.8) represent the system of equations for unknown velocity and vorticity. The only exception is the main term $\widetilde{\boldsymbol{u}}_0$ of oscillatory velocity, which is potential (due to (5.9)), and remains to be prescribed. There are several ways to prescribe it in different problems: (i) it can be forced by oscillating boundary conditions; (ii) it can appear as self-oscillations; (iii) it can be maintained by an external oscillating force; or (iv) it can appear in full viscous theory from the procedure of matching an outer flow with an oscillatory boundary layer solution. Hence two terms of drift velocity $\overline{\boldsymbol{V}}_0$ and $\overline{\boldsymbol{V}}_1$ (3.21) represent the functions that are 'external' for the averaged equations (5.6)-(5.8), while all other terms of drift velocity and all mean-fields (including pseudo-diffusion) are explicitly defined by both $\widetilde{\boldsymbol{u}}_0$ and unknown solutions of (5.6).

6. Distinguished limit

We formulate the idea of a distinguish limit using the problem of *Sect.*3 (scalar transport) for a special (s-independent) non-degenerated velocity field

$$\boldsymbol{u} = \widehat{\boldsymbol{u}}(\boldsymbol{x}, \tau), \quad \widehat{\boldsymbol{u}} \in \mathbb{O}(1) \cap \mathbb{H}, \quad \tau \equiv \sigma t, \quad \varepsilon \equiv 1/\sigma \tag{6.1}$$

Let us select the class of *trial* slow time-variables as $s = \sigma^\alpha t$, where $\alpha = \text{const}$ (it should be $\alpha < 1$ since s-variable must be 'slow' in comparison with $\tau = \sigma t$). Then the two-timing form of eqn. (3.4) is

$$\left(\frac{\partial}{\partial \tau} + \varepsilon^{1-\alpha}\frac{\partial}{\partial s}\right)\widehat{c} + \varepsilon\widehat{\boldsymbol{u}} \cdot \nabla\widehat{c} = 0, \quad \widehat{c} = \widehat{c}(\boldsymbol{x}, s, \tau) \tag{6.2}$$

where τ and s are independent variables. Surprisingly, the slow time-scale (and the value of α) is uniquely determined by the structure of the equation. It is determined as the distinguished limit, which includes the following properties: (i) the solution for $\alpha = \alpha_d$ is given by a valid asymptotic procedure; (ii) all solutions for $\alpha_d < \alpha < 1$ contain terms secular in s, and (iii) for any $\alpha < \alpha_d$ the system of equations for successive approximations contains internal contradictions and it is unsolvable (unless it degenerates). One can show that in all three problems, considered in *Sects.*3-5, $\alpha_d = 0$. We have fully studied the first three approximations for all three problems, therefore it can be accepted that the validity

of the procedures with $\alpha_d = 0$ (item (i) above) has been established. The properties (ii) and (iii) for $\alpha_d = 0$ we accept without proof. The value $\alpha_d = 0$ justifies our choice $T = T_i$ in (3.2).

We should emphasise that the nature of time-variables t and s is different: the 'modulation' variable t in $\widehat{\boldsymbol{u}}(\boldsymbol{x}, t, \omega t)$ (3.5) is given by the original physical formulation of the problem, while slow time-variable s in the equation (3.10) uniquely appears from the distinguished limit consideration. All considerations of *Sects.*3-5 are based on the coincidence (which is the same as $\alpha_d = 0$) of slow time-variable s with physical time t. This lucky coincidence allows us to return from a special formula for velocity (6.1) to the original general functional form (3.5). In more general situations (when the scales s and t have different orders) one should consider the problems with three time-variables: s, t, and τ. In our case $s = t$, therefore we do not need to use the three-timing method.

One more example of a distinguished limit can be seen in the averaged equations (3.18)-(3.20). Let us take a degenerated (predominantly oscillating) flow $\overline{\boldsymbol{u}}_0 \equiv 0$, $\widetilde{\boldsymbol{u}}_0 \not\equiv 0$ and look for a solution with $\alpha = 0$. In this case eqns.(3.18),(3.19) give

$$\overline{c}_{0s} = 0 \tag{6.3}$$

$$\overline{c}_{1s} = -(\overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0) \cdot \nabla \overline{c}_0 \tag{6.4}$$

The integration of the first equation gives $\overline{c}_0$ as an arbitrary function of $\boldsymbol{x}$ only; its substitution into the second equation shows that (in the generic case) $\overline{c}_1$ grows linearly with s. The roots of this trouble lie in the fact that for predominantly oscillating flows $\alpha_d = -1$ and $s = t/\sigma$, see Vladimirov (2011); a wrongly chosen variable s inevitably leads to the appearance of secular terms.

7. Conditional isomorphism between the high-frequency asymptotic solutions and small-amplitude asymptotic solutions

The area of applicability of the derived above results is essentially expanded by the isomorphism between the high-frequency asymptotic solutions and the small-amplitude asymptotic solutions. In order to clarify the existence of such isomorphism, let us rewrite eqn.(3.4) for a newly defined unmodulated velocity $\boldsymbol{u}$

$$\boldsymbol{u} = \varepsilon_1 \widehat{\boldsymbol{u}}_1(\boldsymbol{x}, \tau), \quad \widehat{\boldsymbol{u}}_1 \in \mathbb{O}(1) \cap \mathbb{H}, \quad \tau \equiv \sigma t, \quad \sigma = O(1) \tag{7.1}$$

$$c_\tau + \varepsilon_1 \frac{1}{\sigma} \widehat{\boldsymbol{u}}_1 \cdot \nabla c = 0, \quad \operatorname{div} \widehat{\boldsymbol{u}}_1 = 0, \quad \varepsilon_1 \to 0 \tag{7.2}$$

where ε_1 is a small amplitude of velocity $\boldsymbol{u}$. The most important difference with the case of *Sects.*3-5 is $\sigma = O(1)$. The two-timing form of solution is

$$c = \widehat{c}(\boldsymbol{x}, s_1, \tau_1) \quad \text{where} \quad \tau_1 = \tau; \quad s_1 = \varepsilon_1^\beta \tau; \quad \frac{\partial}{\partial \tau} = \frac{\partial}{\partial \tau_1} + \varepsilon_1^\beta \frac{\partial}{\partial s_1} \tag{7.3}$$

where s_1 is a *trial* slow time-variable with an indefinite constant $\beta > 0$. The substitution of (7.3) into (7.2) gives

$$\left(\frac{\partial}{\partial \tau_1} + \varepsilon_1^\beta \frac{\partial}{\partial s_1} \right) \widehat{c} + \varepsilon_1 \frac{1}{\sigma} \widehat{\boldsymbol{u}}_1 \cdot \nabla \widehat{c} = 0 \tag{7.4}$$

which is our stage to declare that the variable τ_1 and s_1 are temporarily independent (*cf.* with (3.9)). One can see that the mathematical form of eqn.(7.4) coincides with (6.2). Therefore, according to the result of the previous section, the distinguished limit

corresponds to $\beta = 1$ and the final equation to be solved is

$$\widehat{c}_{\tau_1} + \varepsilon_1 \widehat{c}_{s_1} + \varepsilon_1 \frac{1}{\sigma} \widehat{\boldsymbol{u}}_1 \cdot \nabla \widehat{c} = 0 \tag{7.5}$$

The comparison between (7.5) and (3.10) shows the mathematical isomorphism between the high-frequency asymptotic problem and the small-amplitude asymptotic problem, where the replacements

$$\tau \leftrightarrow \tau_1, \quad s \leftrightarrow s_1, \quad \widehat{\boldsymbol{u}} \leftrightarrow \widehat{\boldsymbol{u}}_1/\sigma, \quad \varepsilon \leftrightarrow \varepsilon_1 \tag{7.6}$$

provide the transformation of the equations (7.5) and (3.10) into each other.

We need to add some further comments:

1. The slow variable $s_1 = \varepsilon_1 \sigma t$ is different from t, therefore, in contrast with (3.5), t can not included to the list of original variables of velocity (7.1).

2. The area of applicability of the introduced isomorphism is restricted by velocity fields in the form (7.1). Physically, the expressions in the form (7.1) mean that we are allowed to consider only steady velocity fields $\overline{\boldsymbol{u}}(\boldsymbol{x})$ with superimposed unmodulated oscillations $\widetilde{\boldsymbol{u}}(\boldsymbol{x}, \tau)$. Such functional form can be deliberately chosen for the first two problems (scalar transport and kinematic MHD). For vortex dynamics one should impose the same functional restrictions on $\widehat{\boldsymbol{u}}_0$.

3. The existence of similar to (7.6) isomorphisms should be checked separately for each new equation. In particular, the adding of diffusivity or viscosity can alter the appearance of isomorphism.

4. One can find that the discussed isomorphism can be also established by the simultaneous rescaling of time-variable τ and velocity $\boldsymbol{u}$ in (7.1). We think that our way is preferable since: (i) it preserves the original physical variables; (ii) it can be used in more general equations.

5. The high-frequency problems and small-amplitude problems are physically different. The introduced isomorphism shows only their mathematical equivalence, which is valid under the formulated restrictions.

6. The existence of isomorphism (7.6) explains why some high-frequency solutions and small-amplitude solutions can coincide (or can be close to each other). For example, it explains why the average equation for vorticity by Craik & Leibovich (1976) is very similar to eqn.(5.8) and represents a steady version of the MHD-Drift equation from Vladimirov (2011), taken without magnetic field.

8. On the nature of pseudo-diffusion (PD)

The pseudo-diffusion (PD) term that appears in the averaged equations (3.20), (3.29) might be seen as a physically new result. The prefix *pseudo-* appears in our study due to three reasons: (i) in the successive approximations PD appears for the first time in the equation of the second approximation (not in the zero-order equation); hence, mathematically PD always plays a part of the known RHS in the equation of the second approximation; (ii) we use only regular asymptotic expansions (3.11); and (iii) PD coefficient (matrix) has a specific form of Lie-derivative (3.20),(3.24). The first two reasons are very common (see *e.g.* Magar & Pedley (2005)), they comply with the role of PD as an effective physical diffusion. However, the third reason might lead us to the suggestion that this term gives a s-dependent kinematic deformation of the averaged field, which is not related to physical diffusion. We illustrate the role of this term in a simple example of three-dimensional translational rigid-body oscillations.

Let an infinite fluid oscillate translationally as a rigid-body with a small displacement

$\widetilde{\boldsymbol{x}}(s,\tau)=O(\varepsilon)$, where $\varepsilon=1/\sigma$. Then the Eulerian coordinate of a particle is $\boldsymbol{x}=\overline{\boldsymbol{x}}+\widetilde{\boldsymbol{x}}$. The drift velocity identically vanishes in all orders $\overline{\boldsymbol{V}}\equiv 0$, hence for any particle $\overline{\boldsymbol{x}}\equiv$ const. Therefore, we accept that $\widetilde{\boldsymbol{x}}(0,0)=0$, which means that the Lagrangian (initial) coordinate of each particle coincides with its unperturbed position $\boldsymbol{X}=\overline{\boldsymbol{x}}$. Then the solution is

$$\widehat{c}(\boldsymbol{x},s,\tau)=c^L(\boldsymbol{x}-\widetilde{\boldsymbol{x}}) \tag{8.1}$$

where $c^L(\boldsymbol{X})$ is a given time-independent distribution in Lagrangian coordinates. One can expand both sides of (8.1) as

$$\widehat{c}_0+\varepsilon\widehat{c}_1+\varepsilon^2\widehat{c}_2+\ldots=c^L(\boldsymbol{x})-\widetilde{x}_i\frac{\partial c^L(\boldsymbol{x})}{\partial x_i}+\frac{\varepsilon^2}{2}\widetilde{x}_i\widetilde{x}_k\frac{\partial^2 c^L(\boldsymbol{x})}{\partial x_i\partial x_k}+\ldots \tag{8.2}$$

where the LHS corresponds to the decomposition (3.11) with $\widehat{c}_n=\widehat{c}_n(\boldsymbol{x},s,\tau)=\overline{c}_n(\boldsymbol{x},s)+\widetilde{c}_n(\boldsymbol{x},s,\tau)$, while the RHS represents the Taylor's series. The averaging (2.2) of (8.2) yields

$$\overline{c}_0+\varepsilon\overline{c}_1+\varepsilon^2\overline{c}_2+\ldots=c^L(\boldsymbol{x})+\frac{\varepsilon^2}{2}\langle\widetilde{\xi}_i\widetilde{\xi}_k\rangle\frac{\partial^2 c^L(\boldsymbol{x})}{\partial x_i\partial x_k}+\ldots \tag{8.3}$$

where we have changed $\widetilde{\boldsymbol{x}}(s,\tau)$ to $\varepsilon\widetilde{\boldsymbol{\xi}}(s,\tau)$ (3.21), which is valid with the given precision (by an elementary consideration one can derive that $\widetilde{\boldsymbol{x}}=\varepsilon\widetilde{\boldsymbol{\xi}}-\varepsilon^2\widetilde{\boldsymbol{\xi}}_s^{\tau}+O(\varepsilon^3)$). The s-derivative of (8.3) is

$$\overline{c}_{0s}+\varepsilon\overline{c}_{1s}+\varepsilon^2\overline{c}_{2s}+\ldots=\frac{\varepsilon^2}{2}\langle\widetilde{\xi}_i\widetilde{\xi}_k\rangle_s\frac{\partial^2 \overline{c}_0(\boldsymbol{x})}{\partial x_i\partial x_k}+\ldots \tag{8.4}$$

where we have used $c^L=\overline{c}_0$ (which is the leading term of (8.3)). One can see that eqn.(8.4) coincides with (3.18)-(3.20) taken for $\overline{\boldsymbol{u}}_0\equiv 0$ and for the zero drift.

This example indicates that PD represents a purely kinematic correction which appears due to the averaging operation. In addition, one might put forward a conjecture that all mean-field terms in (3.29),(4.20),(5.13) are also purely kinematic and do not represent any physical processes. However, the same 'philosophical' question arises in the theory of turbulence: let us take an ensemble of flows, which differ from each other only by translations as a whole; then the averaging operation with a time-dependent distribution function leads to 'turbulent diffusion'. Can it be considered physically meaningful? The answer to this question requires further analysis. Anyway, the significance of PD (and other mean-field terms) for the results of this paper lies in the discovery of the universal mathematical structure of the terms, that appear from the straightforward calculations in addition to the drift-type corrections of the averaged equations.

9. Discussion

1. The main novel element of our consideration is a universal structure of averaged equations describing different processes in non-degenerated oscillatory flows. The most useful problem (for the analysis of this structure) is the one for a scalar admixture (*Sect.*3), where the calculations are the simplest and at the same time sufficient enough to produce the analytical expressions and constructions, that appear in all other cases, including predominantly oscillating (degenerated) flows, see Vladimirov (2010), Vladimirov (2011).

2. The averaged equations for a 'frozen-in' magnetic field and vorticity dynamics formally coincide; the difference between two cases is the constraint linking velocity with vorticity in the latter one. However, the presence of such a constraint has entirely changed the nature of the equations: the velocity field is prescribed in the MHD-kinematics but it represents an unknown function in vortex dynamics.

3. For all considered non-degenerated flows a non-linear averaged equation appears only in the mean (zero) approximation of vortex dynamics; its form coincides with that of the original (exact) equations. All other equations are linear. The most impressive in the analytical beauty and demanding physical explanations is an appearance of Lie-derivatives in the pseudo-diffusion matrix and in other mean-fields.

4. In this paper we intensely use the notion of a drift. It is known that a drift velocity can appear from Lagrangian, Eulerian, or hybrid (Euler-Lagrange) considerations. In our study we use the *Eulerian drift*, which appears as a result of the Eulerian averaging of the governing PDEs without direct addressing the motion of particles, see Craik & Leibovich (1976), Craik (1985), Riley (2001), Vladimirov (2010), Vladimirov (2011), Ilin & Morgulis (2011). More classical *Lagrangian drift* appears as the average motion of Lagrangian particles and its theory is based on the averaging of ODEs, see Stokes (1847), Lamb (1932), Longuet-Higgins (1953), Batchelor (1967); the hybrid drift coincides with the Lagrangian one, see Andrews & McIntyre (1978), Craik (1985), Soward & Roberts (2010), Vladimirov (2010). It is known that Lagrangian and Eulerian drifts are similar, but not identical to each other, see Vladimirov (2010).

5. The first two examples (the transport of a scalar and the kinematic MHD) are not always mutually independent: for the translationally-invariant motions eqn.(4.1) can be reduced to (3.1), see Vladimirov, Moffatt and Ilin (1996).

6. The small parameter of our asymptotic theory is inverse dimensionless frequency $\varepsilon = 1/\omega$. One can see that (since the oscillatory velocity is $O(1)$) the dimensionless characteristic oscillatory displacement of a material particle is $O(\varepsilon)$. It is interesting, that for the isomorphic small-amplitude flow of *Sect.*7 this displacement is $O(\varepsilon_1)$.

7. For a finite and time-dependent flow domain $\mathcal{D}(t)$ the definition of average (2.2) works only if $\boldsymbol{x} \in \mathcal{D}$ at any instant. If it is not true, then the theory should include the 'projection' of a boundary condition on an 'undisturbed' boundary. Such a consideration requires the smallness of the oscillatory displacements of fluid particles, which are always proportional to ε (see item 6 above). Hence, the generalizations of all our results to finite domains can be undertaken without adding any new small parameters to the list (3.8).

8. Viscosity or diffusivity can be routinely added to all the above considerations as the RHS-term $\kappa\nabla^2\widehat{a}$, $\nu\nabla^2\widehat{\boldsymbol{u}}$, *etc.* However, the final appearance of such terms depends on the order of magnitude of the new small parameter (like $1/Pe \equiv \kappa/\sigma L^2$, where Pe is the Peclet number) in terms of ε. Different relations between small parameters will produce different averaged equations. The most 'harmless' case of adding viscosity or diffusivity is to take $\kappa/\sigma L^2 = O(\varepsilon^2)$. In this case viscous or diffusion terms will be just added to all the equations of the second approximation. This assumption has been intensely used in biological applications, see Magar & Pedley (2005). It is interesting to note that in such cases dimensionless physical diffusion and pseudo-diffusion are of the same order of magnitude. We avoid viscosity and diffusivity in this paper in order to focus our attention on 'pure' kinematics and dynamics.

9. The incorporation of density stratification and/or compressibility into all considered problems is straightforward. In fact, a similar theory of the predominantly oscillating flows was developed for a compressible fluid, see Vladimirov (2010).

10. In this paper we consider periodic (in fast time-variable τ) functions. The studies of non-periodic in τ oscillations can represent the next stages of research. For example, such a development has been already available for Langmuir circulations, see Craik & Leibovich (1976), Leibovich (1983), Craik (1985).

11. The described transformations make partially visible the same properties of the governing equations of fluid dynamics as were studied by the hybrid Euler-Lagrange GLM-procedure, see Andrews & McIntyre (1978), Buhler (2009), Soward & Roberts (2010).

However, the GLM-method in its original form is not adapted to the two-timing framework: it is sufficient to note that a fast-time variable and a related small parameter (the ratio of two time-scales) do not appear in Andrews & McIntyre (1978). The two-timing adaptation of the GLM-kinematics has been developed by Vladimirov (2010). Further steps in this direction are required to introduce the two-timing dynamics.

12. Degenerated flows with $\overline{\boldsymbol{u}}_0 \equiv 0$ (predominantly oscillating flows) were considered by Stokes (1847), Craik & Leibovich (1976), Riley (2001), Magar & Pedley (2005), Buhler (2009), Vladimirov (2010), Herreman & Lesaffre (2011), Vladimirov (2011), and many others. The results of Vladimirov (2010), Vladimirov (2011) show that the averaged governing equations of degenerated flows contain the same (or similar) building blocks/elements, as the ones presented above. The vortex dynamics of such flows is extremely interesting theoretically, since it produces non-linear equations for averaged vorticity, that include a drift in their coefficients. A classical application of this theory is the describing of the Langmuir circulations, see Craik & Leibovich (1976), Leibovich (1983), Thorpe (2004).

13. There are many areas where the results of *Sects.*3-7 could be applied, such as waves in a flow with strong shear, flows past translationally moving and oscillating solids, oscillatory flows in channels and pipes, various acoustical problems, *etc.* The number of applications can be very large. However we do not consider any applications, since: (i) the focus of this paper is the universal structure of the averaged governing equations; (ii) every application of a general theory requires its own paper.

14. This paper is written as Part 1 of a series, where Part 2 (a similar theory for degenerated flows) is based on Vladimirov (2010). A number of examples illustrating the values of drift and pseudo-diffusion is given in Part 2, since predominantly oscillating flows have many well-developed applications.

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Appendix A. Deriving of the equations for a passive admixture

The equation of zero approximation (3.13) is

$$\widehat{c}_{0\tau} = \widetilde{c}_{0\tau} = 0 \tag{A 1}$$

Its unique solution is $\widetilde{c}_0 \equiv 0$ (2.7), while $\overline{c}_0$ remains undetermined; let us write it as

$$\widetilde{c}_0 \equiv 0, \quad \overline{c}_0 = \boxed{?} \tag{A 2}$$

where the first equality gives (3.25). The equation of the first approximation (3.14) (with the use of (A 2)) is:

$$\widetilde{c}_{1\tau} + \widehat{\boldsymbol{u}}_0 \cdot \nabla\overline{c}_0 + \overline{c}_{0s} = 0 \tag{A 3}$$

Its bar-part (2.4) and tilde-part (2.3) are

$$\overline{c}_{0s} + \overline{\boldsymbol{u}}_0 \cdot \nabla\overline{c}_0 = 0 \tag{A 4}$$

$$\widetilde{c}_{1\tau} + \widetilde{\boldsymbol{u}}_0 \cdot \nabla\overline{c}_0 = 0 \tag{A 5}$$

where (A 4) represents the final equation for $\overline{c}_0$ (3.18). The $\mathbb{T}$-integration (2.6) of (A 5) yields (3.26) and keeps $\overline{c}_1$ unknown

$$\widetilde{c}_1 = -\widetilde{\boldsymbol{u}}_0^{\tau} \cdot \nabla\overline{c}_0, \quad \overline{c}_1 = \boxed{?} \tag{A 6}$$

The equation of second approximation (3.15) that takes into account (A 2) is

$$\widetilde{c}_{2\tau} + \widehat{\boldsymbol{u}}_1 \cdot \nabla \overline{c}_0 + \widehat{\boldsymbol{u}}_0 \cdot \nabla \widehat{c}_1 + \widehat{c}_{1s} = 0 \tag{A 7}$$

Its bar-part and tilde-part are

$$\overline{c}_{1s} + \overline{\boldsymbol{u}}_1 \cdot \nabla \overline{c}_0 + \overline{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_1 + \langle \widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_1 \rangle = 0 \tag{A 8}$$

$$\widetilde{c}_{2\tau} + \widetilde{\boldsymbol{u}}_1 \cdot \nabla \overline{c}_0 + \overline{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_1 + \widetilde{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_1 + \{\widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_1\} + \widetilde{c}_{1s} = 0 \tag{A 9}$$

where $\langle\cdot\rangle$ stands for the average (2.2) and $\{\cdot\}$ for the tilde-part (2.8). Eqn. (A 8) can be transformed (with the use of (A 6) and (2.10)-(2.16)) into the form (3.19)

$$\overline{c}_{1s} + \overline{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_1 + (\overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0) \cdot \nabla \overline{c}_0 = 0 \tag{A 10}$$

$$\overline{\boldsymbol{V}}_0 \equiv \frac{1}{2}\langle [\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{u}}_0^\tau] \rangle \tag{A 11}$$

while the $\mathbb{T}$-integration (2.6) of (A 9) gives (3.27)

$$\widetilde{c}_2 = -\widetilde{\boldsymbol{u}}_1^\tau \cdot \nabla \overline{c}_0 - \overline{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_1^\tau - \widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla \overline{c}_1 - \{\widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_1\}^\tau - \widetilde{c}_{1s}^\tau, \quad \overline{c}_2 = \boxed{?} \tag{A 12}$$

The equation of third approximation (3.16) that takes into account (A 2) is

$$\widetilde{c}_{3\tau} + \widehat{\boldsymbol{u}}_2 \cdot \nabla \overline{c}_0 + \widehat{\boldsymbol{u}}_0 \cdot \nabla \widehat{c}_2 + \widehat{\boldsymbol{u}}_1 \cdot \nabla \widehat{c}_1 + \widehat{c}_{2s} = 0 \tag{A 13}$$

Its bar-part and tilde-part are

$$\overline{c}_{2s} + \overline{\boldsymbol{u}}_2 \cdot \nabla \overline{c}_0 + \overline{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_2 + \overline{\boldsymbol{u}}_1 \cdot \nabla \overline{c}_1 + \langle \widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_2 \rangle + \langle \widetilde{\boldsymbol{u}}_1 \cdot \nabla \widetilde{c}_1 \rangle = 0 \tag{A 14}$$

$$\begin{aligned} &\widetilde{c}_{3\tau} + \widetilde{\boldsymbol{u}}_2 \cdot \nabla \overline{c}_0 + \widetilde{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_2 + \overline{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_2 + \widetilde{\boldsymbol{u}}_1 \cdot \nabla \overline{c}_1 + \overline{\boldsymbol{u}}_1 \cdot \nabla \widetilde{c}_1 + \\ &\{\widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_2\} + \{\widetilde{\boldsymbol{u}}_1 \cdot \nabla \widetilde{c}_1\} + \widetilde{c}_{2s} = 0 \end{aligned} \tag{A 15}$$

We transform the 'Reynolds stress' terms in (A 14) with the use of (A 6),(A 12),(2.10)-(2.16)

$$R_c \equiv \langle \widetilde{\boldsymbol{u}}_1 \cdot \nabla \widetilde{c}_1 \rangle + \langle \widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_2 \rangle = R_{c1} + R_{c2} + R_{c3} + R_{c4} \tag{A 16}$$

$$R_{c1} \equiv -\langle (\widetilde{\boldsymbol{u}}_0 \cdot \nabla)(\widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla) \rangle \overline{c}_1 = \overline{\boldsymbol{V}}_0 \cdot \nabla \overline{c}_1 \tag{A 17}$$

$$R_{c2} \equiv -\langle (\widetilde{\boldsymbol{u}}_1 \cdot \nabla)(\widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla) \rangle \overline{c}_0 - \langle (\widetilde{\boldsymbol{u}}_0 \cdot \nabla)(\widetilde{\boldsymbol{u}}_1^\tau \cdot \nabla) \rangle \overline{c}_0 = \overline{\boldsymbol{V}}_{01} \cdot \nabla \overline{c}_0 \tag{A 18}$$

$$\begin{aligned} &R_{c3} \equiv -\langle (\widetilde{\boldsymbol{u}}_0 \cdot \nabla)\{(\widetilde{\boldsymbol{u}}_0 \cdot \nabla)\widetilde{c}_1\}^\tau \rangle = \langle (\widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla)(\widetilde{\boldsymbol{u}}_0 \cdot \nabla)\widetilde{c}_1 \rangle = \\ &-\langle (\widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla)(\widetilde{\boldsymbol{u}}_0 \cdot \nabla)(\widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla) \rangle \overline{c}_0 = \overline{\boldsymbol{V}}_1 \cdot \nabla \overline{c}_0 \end{aligned} \tag{A 19}$$

$$\begin{aligned} &R_{c4} \equiv -\langle (\widetilde{\boldsymbol{u}}_0 \cdot \nabla)\widetilde{c}_{1s}^\tau \rangle - \langle (\widetilde{\boldsymbol{u}}_0 \cdot \nabla)(\overline{\boldsymbol{u}}_0 \cdot \nabla)\widetilde{c}_1^\tau \rangle = \\ &\langle (\widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla) D_0 \widetilde{c}_1 \rangle = \overline{\boldsymbol{V}}_\chi \cdot \nabla \overline{c}_0 - \frac{\partial}{\partial x_i}\left(\overline{\chi}_{ik}\frac{\partial \overline{c}_0}{\partial x_k}\right) \end{aligned} \tag{A 20}$$

$$\overline{\boldsymbol{V}}_0 \equiv \frac{1}{2}\langle [\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{\xi}}] \rangle, \quad \overline{\boldsymbol{V}}_1 \equiv \frac{1}{3}\langle [[\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{\xi}}], \widetilde{\boldsymbol{\xi}}] \rangle, \quad \widetilde{\boldsymbol{\xi}} \equiv \widetilde{\boldsymbol{u}}_0^\tau \tag{A 21}$$

$$\overline{\boldsymbol{V}}_{10} \equiv \langle [\widetilde{\boldsymbol{u}}_1, \widetilde{\boldsymbol{\xi}}] \rangle, \quad \boldsymbol{V}_\chi \equiv \frac{1}{2}\langle [\widetilde{\boldsymbol{\xi}}, \mathfrak{L}\widetilde{\boldsymbol{\xi}}] \rangle, \quad \overline{\chi}_{ik} \equiv \frac{1}{2}\mathfrak{L}\langle \widetilde{\xi}_i \widetilde{\xi}_k \rangle, \tag{A 22}$$

where the operator $\mathfrak{L}$ (Lie-derivative) is such that its action on $\widetilde{\boldsymbol{\xi}}$ and $\langle \widetilde{\xi}_i \widetilde{\xi}_k \rangle$ is defined as

$$\mathfrak{L}\widetilde{\boldsymbol{\xi}} \equiv \widetilde{\boldsymbol{\xi}}_s + [\widetilde{\boldsymbol{\xi}}, \overline{\boldsymbol{u}}_0] \tag{A 23}$$

$$\mathfrak{L}\langle \widetilde{\xi}_i \widetilde{\xi}_k \rangle \equiv (\partial_s + \overline{\boldsymbol{u}}_0 \cdot \nabla)\langle \widetilde{\xi}_i \widetilde{\xi}_k \rangle - \frac{\partial \overline{u}_{0k}}{\partial x_m}\langle \widetilde{\xi}_i \widetilde{\xi}_m \rangle - \frac{\partial \overline{u}_{0i}}{\partial x_m}\langle \widetilde{\xi}_k \widetilde{\xi}_m \rangle \tag{A 24}$$

Hence eqn.(A 14) takes a form

$$\overline{c}_{2s} + \overline{\boldsymbol{u}}_0 \cdot \nabla \overline{c}_2 + (\overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0) \cdot \nabla \overline{c}_1 + \\ (\overline{\boldsymbol{u}}_2 + \overline{\boldsymbol{V}}_{10} + \overline{\boldsymbol{V}}_1 + \overline{\boldsymbol{V}}_\chi) \cdot \nabla \overline{c}_0 = \frac{\partial}{\partial x_i}\left(\overline{\chi}_{ik}\frac{\partial \overline{a}_0}{\partial x_k}\right) \tag{A 25}$$

while (A 15) can be $\mathbb{T}$-integrated (2.6)

$$\widetilde{c}_{3\tau} = -\widetilde{\boldsymbol{u}}_2^\tau \cdot \nabla \overline{c}_0 - \widetilde{\boldsymbol{u}}_0^\tau \cdot \nabla \overline{c}_2 - \overline{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_2^\tau - \widetilde{\boldsymbol{u}}_1^\tau \cdot \nabla \overline{c}_1 - \overline{\boldsymbol{u}}_1 \cdot \nabla \widetilde{c}_1^\tau - \\ \{\widetilde{\boldsymbol{u}}_0 \cdot \nabla \widetilde{c}_2\}^\tau - \{\widetilde{\boldsymbol{u}}_1 \cdot \nabla \widetilde{c}_1\}^\tau + \widetilde{c}_{2s}^\tau, \quad \overline{c}_3 = \boxed{?} \tag{A 26}$$

Appendix B. Deriving of equations for kinematic MHD

The equations of the zeroth approximation is

$$\widehat{\boldsymbol{h}}_{0\tau} = \widetilde{\boldsymbol{h}}_{0\tau} = 0 \tag{B 1}$$

Its unique solution is $\widetilde{\boldsymbol{h}}_0 \equiv 0$ (2.7), while $\overline{\boldsymbol{h}}_0$ remains undetermined; let us write it as

$$\widetilde{\boldsymbol{h}}_0 \equiv 0, \quad \overline{\boldsymbol{h}}_0 = \boxed{?} \tag{B 2}$$

The equation of the first approximation of (4.5) (with the use of (B 2)) is:

$$\widetilde{\boldsymbol{h}}_{1\tau} + [\,\overline{\boldsymbol{h}}_0, \widehat{\boldsymbol{u}}_0] + \overline{\boldsymbol{h}}_{0s} = 0 \tag{B 3}$$

Its bar-part (2.4) and tilde-part (2.3) are

$$\overline{\boldsymbol{h}}_{0s} + [\,\overline{\boldsymbol{h}}_0, \overline{\boldsymbol{u}}_0] = 0 \tag{B 4}$$

$$\widetilde{\boldsymbol{h}}_{1\tau} + [\,\overline{\boldsymbol{h}}_0, \widetilde{\boldsymbol{u}}_0] = 0 \tag{B 5}$$

Eqn.(B 4) represents a final equation for $\overline{\boldsymbol{h}}_0$. The $\mathbb{T}$-integration (2.6) of (B 5) yields

$$\widetilde{\boldsymbol{h}}_1 = [\widetilde{\boldsymbol{u}}_0^\tau, \overline{\boldsymbol{h}}_0], \quad \overline{\boldsymbol{h}}_1 = \boxed{?} \tag{B 6}$$

The equation of the second approximation that takes into account (B 2) is

$$\widetilde{\boldsymbol{h}}_{2\tau} + \widehat{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_0, \widehat{\boldsymbol{u}}_1] + [\widehat{\boldsymbol{h}}_1, \widehat{\boldsymbol{u}}_0] = 0 \tag{B 7}$$

Its bar-part and tilde-part are

$$\overline{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_0, \overline{\boldsymbol{u}}_1] + [\overline{\boldsymbol{h}}_1, \overline{\boldsymbol{u}}_0] + \langle[\widetilde{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0]\rangle = 0 \tag{B 8}$$

$$\widetilde{\boldsymbol{h}}_{2\tau} + \widetilde{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_0, \widetilde{\boldsymbol{u}}_1] + [\widetilde{\boldsymbol{h}}_1, \overline{\boldsymbol{u}}_0] + [\overline{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0] + \{[\widetilde{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0]\} = 0 \tag{B 9}$$

where $\langle\cdot\rangle$ stands for the average operation (2.2) and $\{\cdot\}$ for the tilde-part (2.8). The Reynolds-stress-type term in (B 8) can be transformed with the use of (B 6) and (2.10)-(2.16) as

$$\langle[\widetilde{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0]\rangle = \langle[[\widetilde{\boldsymbol{u}}_0^\tau \overline{\boldsymbol{h}}_0], \widetilde{\boldsymbol{u}}_0]\rangle = [\overline{\boldsymbol{h}}_0, \overline{\boldsymbol{V}}_0] \\ \overline{\boldsymbol{V}}_0 \equiv \frac{1}{2}\langle[\widetilde{\boldsymbol{u}}_0, \widetilde{\boldsymbol{u}}_0^\tau]\rangle \tag{B 10}$$

Hence (B 8) takes a form

$$\overline{\boldsymbol{h}}_{1s} + [\overline{\boldsymbol{h}}_0, \overline{\boldsymbol{u}}_1 + \overline{\boldsymbol{V}}_0] + [\overline{\boldsymbol{h}}_1, \overline{\boldsymbol{u}}_0] = 0 \tag{B 11}$$

while (B 10) can be $\mathbb{T}$-integrated (2.6)

$$\widetilde{\boldsymbol{h}}_2 = -\widetilde{\boldsymbol{h}}_{1s}^\tau - [\overline{\boldsymbol{h}}_0, \widetilde{\boldsymbol{u}}_1^\tau] - [\widetilde{\boldsymbol{h}}_1^\tau, \overline{\boldsymbol{u}}_0] - [\overline{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0^\tau] - \{[\widetilde{\boldsymbol{h}}_1, \widetilde{\boldsymbol{u}}_0]\}^\tau, \quad \overline{\boldsymbol{h}}_2 = \boxed{?} \tag{B 12}$$

The equation of the third approximation that takes into account (B 2) is

$$\widetilde{\boldsymbol{h}}_{3\tau}+\widehat{\boldsymbol{h}}_{2s}+[\overline{\boldsymbol{h}}_0,\widehat{\boldsymbol{u}}_2]+[\widehat{\boldsymbol{h}}_2,\widehat{\boldsymbol{u}}_0]+[\widehat{\boldsymbol{h}}_1,\widehat{\boldsymbol{u}}_1]=0 \tag{B 13}$$

Its bar-part and tilde-part are

$$\overline{\boldsymbol{h}}_{2s}+[\overline{\boldsymbol{h}}_0,\overline{\boldsymbol{u}}_2]+[\overline{\boldsymbol{h}}_2,\overline{\boldsymbol{u}}_0]+[\overline{\boldsymbol{h}}_1,\overline{\boldsymbol{u}}_1]+\langle[\widetilde{\boldsymbol{h}}_2,\widetilde{\boldsymbol{u}}_0]\rangle+\langle[\widetilde{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_1]\rangle=0 \tag{B 14}$$

$$\begin{aligned}&\widetilde{\boldsymbol{h}}_{3\tau}+\widetilde{\boldsymbol{h}}_{2s}+[\overline{\boldsymbol{h}}_0,\widetilde{\boldsymbol{u}}_2]+[\widetilde{\boldsymbol{h}}_2,\overline{\boldsymbol{u}}_0]+[\overline{\boldsymbol{h}}_2,\widetilde{\boldsymbol{u}}_0]+\\&[\widetilde{\boldsymbol{h}}_1,\overline{\boldsymbol{u}}_1]+[\overline{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_1]+\{[\widetilde{\boldsymbol{h}}_2,\widetilde{\boldsymbol{u}}_0]\}+\{[\widetilde{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_1]\}=0\end{aligned} \tag{B 15}$$

We transform the Reynolds-stress-type terms in (B 14) with the use of (B 6),(B 12),(2.10)-(2.16)

$$R_h\equiv\langle[\widetilde{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_1]\rangle+\langle[\widetilde{\boldsymbol{h}}_2,\widetilde{\boldsymbol{u}}_0]\rangle=R_{h1}+R_{h2}+R_{h3}+R_{h4} \tag{B 16}$$

$$R_{h1}\equiv-\langle[[\overline{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_0^{\tau}],\widetilde{\boldsymbol{u}}_0]\rangle=[\overline{\boldsymbol{h}}_1,\overline{\boldsymbol{V}}_0] \tag{B 17}$$

$$R_{h2}\equiv-\langle[[\overline{\boldsymbol{h}}_0,\widetilde{\boldsymbol{u}}_0^{\tau}],\widetilde{\boldsymbol{u}}_1]\rangle-\langle[[\overline{\boldsymbol{h}}_0,\widetilde{\boldsymbol{u}}_1^{\tau}],\widetilde{\boldsymbol{u}}_0]\rangle=[\overline{\boldsymbol{h}}_0,\overline{\boldsymbol{V}}_{01}] \tag{B 18}$$

$$R_{h3}\equiv-\langle[\{[\widetilde{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_0^{\tau}]\}^{\tau},\widetilde{\boldsymbol{u}}_0]\rangle=[\overline{\boldsymbol{h}}_0,\overline{\boldsymbol{V}}_1] \tag{B 19}$$

$$-\langle(\widetilde{\boldsymbol{u}}_0^{\tau}\cdot\nabla)(\widetilde{\boldsymbol{u}}_0\cdot\nabla)(\widetilde{\boldsymbol{u}}_0^{\tau}\cdot\nabla)\rangle\overline{c}_0=\overline{\boldsymbol{V}}_1\cdot\nabla\overline{c}_0 \tag{B 20}$$

$$\begin{aligned}&R_{h4}\equiv-\langle[\widetilde{\boldsymbol{h}}_1^{\tau},\widetilde{\boldsymbol{u}}_0]\rangle-\langle[[\widetilde{\boldsymbol{h}}_1^{\tau},\overline{\boldsymbol{u}}_0],\widetilde{\boldsymbol{u}}_0]\rangle=\langle[\mathfrak{L}\widetilde{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_0^{\tau}]\rangle=\\&[\overline{\boldsymbol{h}}_0,\overline{\boldsymbol{V}}_{\chi}]-\frac{\partial}{\partial x_k}\left(\overline{\chi}_{kl}\frac{\partial\overline{h}_{0i}}{\partial x_l}\right)+2\overline{b}_{ikl}\frac{\partial\overline{h}_{0l}}{\partial x_k}+\overline{a}_{ik}\overline{h}_k\end{aligned} \tag{B 21}$$

where the additional to (A 21),(A 22) notations are:

$$\overline{b}_{ikl}\equiv\mathfrak{L}\left\langle\widetilde{\xi}_k\frac{\partial\widetilde{\xi}_i}{\partial x_l}\right\rangle,\quad \overline{a}_{ik}\equiv\mathfrak{L}\left\langle\widetilde{\xi}_l\frac{\partial^2\widetilde{\xi}_i}{\partial x_l\partial x_k}-\frac{\partial\widetilde{\xi}_l}{\partial x_k}\frac{\partial\widetilde{\xi}_i}{\partial x_l}\right\rangle \tag{B 22}$$

Hence eqn.(A 14) takes a form (with the i-component in the LHS)

$$\begin{aligned}&\overline{\boldsymbol{h}}_{2s}+[\overline{\boldsymbol{h}}_2,\overline{\boldsymbol{u}}_0]+[\overline{\boldsymbol{h}}_1,(\overline{\boldsymbol{u}}_1+\overline{\boldsymbol{V}}_0)]+[\overline{\boldsymbol{h}}_0,(\overline{\boldsymbol{u}}_2+\overline{\boldsymbol{V}}_{10}+\overline{\boldsymbol{V}}_1+\overline{\boldsymbol{V}}_{\chi})]\big|_i=\\&\frac{\partial}{\partial x_l}\left(\overline{\chi}_{lk}\frac{\partial\overline{h}_{0i}}{\partial x_k}\right)-\overline{b}_{ikl}\frac{\partial\overline{h}_{0l}}{\partial x_k}-\overline{a}_{ik}\overline{h}_k\end{aligned} \tag{B 23}$$

while (B 15) can be $\mathbb{T}$-integrated (2.6)

$$\begin{aligned}&\widetilde{\boldsymbol{h}}_3=-\widetilde{\boldsymbol{h}}_{2s}^{\tau}-[\overline{\boldsymbol{h}}_0,\widetilde{\boldsymbol{u}}_2^{\tau}]-[\widetilde{\boldsymbol{h}}_2^{\tau},\overline{\boldsymbol{u}}_0]-[\overline{\boldsymbol{h}}_2,\widetilde{\boldsymbol{u}}_0^{\tau}]-\\&[\widetilde{\boldsymbol{h}}_1^{\tau},\overline{\boldsymbol{u}}_1]-[\overline{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_1^{\tau}]-\{[\widetilde{\boldsymbol{h}}_2,\widetilde{\boldsymbol{u}}_0]\}^{\tau}-\{[\widetilde{\boldsymbol{h}}_1,\widetilde{\boldsymbol{u}}_1]\}^{\tau},\quad \overline{\boldsymbol{h}}_3=\boxed{?}\end{aligned} \tag{B 24}$$

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